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ORIGINAL PAPER

The variational iteration method for solving the Volterra integro-differential forms of the Lane–Emden equations of the first and the second kind

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Abstract In this paper, we establish the Volterra integro-differential forms of the Lane–Emden equations. We use the variational iteration method (VIM) to effectively treat these forms. The Volterra integro-differential forms of the Lane–Emden equations overcome the singular behavior at the origin $x = 0$ and do not use a variety of Lagrange multipliers. Several numerical examples are examined to show the validity of the integro-differential forms.

Keywords Volterra integro-differential equations · Lane–Emden equation · variational iteration method · singular behavior · Lagrange multipliers

1 Introduction

In this work we will study the Lane–Emden equation of the first kind [1–10]

$$y'' + \frac{k}{x}y' + y^m = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad k > 1. \tag{1}$$

Moreover, we will examine the Lane–Emden equation of the second kind of the form [11–20]

$$y'' + \frac{k}{x}y' + e^y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad k \geq 1. \tag{2}$$

Equation (1) is the singular Lane–Emden equation of the first kind, or of index m. This equation is a basic equation in the theory of stellar structure [2]. It is used in

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astrophysics for computing the structure of interiors of polytropic stars. This equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of thermodynamics [2,17–19]. In addition, the Lane–Emden equation of the first kind appears also in other context such as in the case of radiatively cooling, self-gravitating gas clouds, in the mean-field treatment of a phase transition in critical adsorption and in the modeling of clusters of galaxies [17–19].

In [2], it was shown that closed form non-singular solutions exist for (1) when $m = 0, 1, 5$. However, for other values of m, we obtain only solutions in the form of power series. Notice that this equation is linear for $m = 0, 1$ and nonlinear otherwise.

It is interesting to point out that the Lane–Emden equation of the first kind (1) has an exact solution for $m = 5$ of the form

$$u(x) = \pm \frac{1}{\sqrt{2x}}, \tag{3}$$

which is singular [8] at $x = 0$. This singular solution belongs to a larger class of singular solutions [8] valid for all Lane–Emden equations with $m > 3$ given by

$$u(x) = \left(\frac{2(m-3)}{(m-1)^2 x^2} \right)^{\frac{1}{m-1}}. \tag{4}$$

Srivista [9] found another solution for $m = 5$ that can be written in a compact form, namely

$$u(x) = \pm \frac{\sin\left(\ln \sqrt{x}\right)}{\sqrt{3x - 2x \sin^2\left(\ln \sqrt{x}\right)}}, \tag{5}$$

where the scaling transformation gives a whole family of solutions. Note that this solution is also singular at $x = 0$, but it can be used in composite stellar models [8]. Moreover, for $m = 5$, Goenner and Havas [10] found the following expression for another class of solutions

$$u(x) = \frac{c_1}{\sqrt{-\frac{x}{3} + x \wp\left(\frac{\ln(Bx)}{2}; \frac{4}{3}, -\frac{8}{27} + \frac{16c_1^4}{3} \right)}}, \tag{6}$$

where $\wp$ denotes the Weierstrass elliptic function, and B and c_1 are integration constants.

Equation (2) is the Lane–Emden equation of the second kind that models the non-dimensional density distribution $y(x)$ in an isothermal gas sphere [9]. In the study of stellar structures one considers the star as a gaseous sphere in thermodynamic and hydrostatic equilibrium for a certain equation of state [10]. Using thetransformation

$$y \to -y \tag{7}$$

to transforms the Lane–Emden equation of the second kind (2) into

$$y'' + \frac{2}{x}y' - e^{-y} = 0, \quad y(0) = y'(0) = 0, \tag{8}$$

which is capable of describing what is now commonly known as Bonnor–Ebert gas spheres [9]. However, Richardson [19] used the transformation

$$x \to ix, \quad y \to -y \tag{9}$$

to transform (2) into

$$y'' + \frac{2}{x}y' + e^{-y} = 0, \quad y(0) = y'(0) = 0, \tag{10}$$

that is used in Richardson's theory [19] of thermionic currents, which is related to the emission of electricity from hot bodies.

The Lane–Emden equation was first studied by astrophysicists Jonathan Homer Lane and Robert Emden, where they considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [17–19]. The well-known Lane–Emden equation has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, the theory of thermionic currents, and in the modeling of clusters of galaxies. A substantial amount of work has been done on these types of problems for various structures. The singular behavior that occurs at $x = 0$ is the main difficulty of Eqs. (1)–(2).

Our aim in this work is to establish Volterra integro-differential equation equivalent to the Lane–Emden equation of any kind. The newly established Volterra integro-differential equation will be treated by using the variational iteration method (VIM). We will show that using the integro-differential forms facilitate the computational work and overcomes the singular behavior at $x = 0$. Moreover, the new integro-differential forms do not require a variety of Lagrange multipliers as we noticed in our previous work in [6]. Only we use the Lagrange multiplier $\lambda = -1$ for all cases of the shape factor k. The variational iteration method (VIM) is a well-known systematic method for solving linear and nonlinear equations, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations, etc.

2 Volterra integro-differential form of the Lane–Emden equation

In this section, we will first derive an integro-differential form equivalent to the singular Lane–Emden equation with a shape factor of two. We will further derive the integro-differential form for the generalized Lane–Emden equation of a generalized shape factor of k for $k \geq 1$. We point out here that in another work [7], Volterra integral forms were established for the Lane–Emden equation and the new forms were treated by using the Adomian decomposition method [21–28].

2.1 The Lane–Emden equation of shape factor of 2

The Lane–Emden equation of shape factor of 2 is

$$y'' + \frac{2}{x}y' + f(y) = 0, \quad y(0) = \alpha, \quad y'(0) = 0, \tag{11}$$

where $f(y)$ can take linear or nonlinear forms.

To convert (11) to a Volterra integro-differential form, we first set

$$y(x) = \alpha - \int_0^x t\left(1 - \frac{t}{x}\right) f(y(t))\,dt. \tag{12}$$

Differentiating (12) twice, using the Leibniz rule, gives

$$\begin{aligned} y'(x) &= -\int_0^x \left(\frac{t^2}{x^2}\right) f(y(t))\,dt, \\ y''(x) &= -f(y(x)) + \int_0^x \left(2\frac{t^2}{x^3}\right) f(y(t))\,dt. \end{aligned} \tag{13}$$

Multiplying y' by $\frac{2}{x}$ and adding the result to $y''(x)$ gives the Lane–Emden equation (11). This shows that the Volterra integro-differential form of the Lane–Emden equation (11) of shape factor $k = 2$ is given by

$$y'(x) = -\int_0^x \left(\frac{t^2}{x^2}\right) f(y(t))\,dt,\, y(0) = \alpha. \tag{14}$$

As stated earlier, the Volterra integral form (12) was treated in [7] by using the ADM.

2.2 The Lane–Emden equation of shape factor of $k \geq 1$

The generalized Lane–Emden equation of shape factor of $k \geq 1$ reads

$$y'' + \frac{k}{x}y' + f(y) = 0, \quad y(0) = \alpha, \quad y'(0) = 0, k > 1, \tag{15}$$

where $f(y)$ can take any linear or nonlinear forms.

To convert (15) to an integro-differential form, we first set

$$y(x) = \alpha - \frac{1}{k-1} \int_0^x t \left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f(y(t))\, dt. \tag{16}$$

Differentiating (16) twice, using the Leibniz rule, gives

$$\begin{aligned} y'(x) &= -\int_0^x \left(\frac{t^k}{x^k}\right) f(y(t))\, dt, \\ y''(x) &= -f(y(x)) + \int_0^x k \left(\frac{t^k}{x^{k+1}}\right) f(y(t))\, dt. \end{aligned} \tag{17}$$

Multiplying y' by $\frac{k}{x}$ and adding the result to $y''(x)$ gives the generalized Lane–Emden equation (15). This shows that the Volterra integro-differential form equivalent to the generalized Lane–Emden equation (15) is given by

$$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f(y(t))\, dt, \quad k \geq 1, \quad y(0) = \alpha. \tag{18}$$

Recall that the case of the Volterra integral form (16) was treated before in [7] by using the ADM.

3 Analysis of the method

In what follows, we give a brief presentation of the variational iteration method. The details of this method are now well known and widely applied in the literature, see for example [24–26].

Consider the differential equation

$$Ly + Ny = g(t), \tag{19}$$

where L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term.

To use the VIM, a correction functional for Eq. (19) should be used in the form

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda \left(Ly_n(\xi) + N\tilde{y}_n(\xi) - g(\xi)\right) d\xi, \tag{20}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{y}_n$ as a restricted variation which means $\delta\tilde{y}_n = 0$.

We can easily show that the Lagrange multiplier for the integro-differential equation (18) is given by

$$\lambda = -1. \tag{21}$$

The successive approximations $y_{n+1}, n \geq 0$ of the solution $y(x)$ will be readily obtained upon using any selective function $y_0(x)$. Consequently, the solution

$$y(x) = \lim_{n\to\infty} y_n(x). \tag{22}$$

In other words, the correction functional (20) will give several approximations, and therefore the exact solution is obtained at the limit of the resulting successive approximations.

The Lane–Emden equation of the first kind, or of the second kind, of shape factor k is

$$y'' + \frac{k}{x}y' + f(y) = 0, \quad k \geq 1, y(0) = \alpha, \quad y'(0) = 0. \tag{23}$$

This equation can be transformed, by using (18) to the integro-differential equation

$$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f(y(t))dt, \quad k \geq 1, y(0) = \alpha. \tag{24}$$

For Eq. (24), the correction functional reads

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(t) + \int_0^t \frac{r^k}{t^k} f(r)\, dr\right) dt, \quad y(0) = \alpha, \tag{25}$$

where we used the Lagrange multiplier $\lambda = -1$. Unlike the works in [6], where a variety of distinct Lagrange multipliers were derived depending on the value of the shape factor k, the Lagrange multiplier for Eq. (24) does not depend on the shape factor k and found to be $\lambda = -1$.

4 Applications

In what follows we will select some numerical examples and apply the integro-differential equation form. The first two examples will include the Lane–Emden equation of the first and the second kind for general shape factor $k \geq 1$.

Example 1 We first start with the generalized Lane–Emden equation of the first kind

$$y'' + \frac{k}{x}y' + y^m = 0, \quad k \geq 1, \quad y(0) = 1, \quad y'(0) = 0. \tag{26}$$

This equation can be transformed, by using (26) to the integro-differential equation

$$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) y^m(t)dt, \quad k \geq 1, \quad y(0) = 1. \tag{27}$$

For Eq. (27), the correction functional reads

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(t) + \int_0^t \frac{r^k}{t^k}\, y^m(r)\, dr \right) dt, \quad y(0) = 1, \tag{28}$$

By selecting the zeroth selection $y_0(x) = 1$, and using (28) we list the first few calculated solution components

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1 - \frac{x^2}{2(1+k)},\\
y_2(x) &= 1 - \frac{x^2}{2(1+k)} + \frac{mx^4}{8(1+k)(3+k)},\\
y_3(x) &= 1 - \frac{x^2}{2(1+k)} + \frac{mx^4}{8(1+k)(3+k)} - \frac{m(-3+4m+k(-1+2m))x^6}{48(1+k)^2(3+k)(5+k)},\\
y_4(x) &= 1 - \frac{x^2}{2(1+k)} + \frac{mx^4}{8(1+k)(3+k)} - \frac{m(-3+4m+k(-1+2m))x^6}{48(1+k)^2(3+k)(5+k)}\\
&\quad + \frac{m\left(30-63m+34m^2+k^2\left(2-7m+6m^2\right)+2k\left(8-23m+16m^2\right)\right)x^8}{384(1+k)^3(3+k)(5+k)(7+k)},\\
&\quad \ldots.
\end{aligned}$$

Consequently, the generalized solution takes the form

$$\begin{aligned}
y(x) &= 1 - \frac{1}{2(1+k)}x^2 + \frac{m}{8(1+k)(3+k)}x^4 - \frac{m(-3+4m+k(-1+2m))}{48(1+k)^2(3+k)(5+k)}x^6\\
&\quad + \frac{m\left(30-63m+34m^2+k^2\left(2-7m+6m^2\right)+2k\left(8-23m+16m^2\right)\right)}{384(1+k)^3(3+k)(5+k)(7+k)}x^8 + \cdots,
\end{aligned} \tag{29}$$

that works for $k \geq 1$.

The following exact solutions

$$\begin{aligned}
y(x) &= 1 - \frac{1}{3!}x^2,\\
y(x) &= \frac{\sin x}{x},\\
y(x) &= \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}},
\end{aligned} \tag{30}$$

are obtained for $m = 0$, 1 and 5, respectively and for $k = 2$. These are the non-singular exact solutions of the Lane–Emden equation.

Example 2 In this section, we will study the Lane–Emden equation of the second kind

$$y'' + \frac{k}{x}y' + e^y = 0, \quad y(0) = y'(0) = 0, \quad k \geq 1. \tag{31}$$

This equation, for $k = 2$, models the distribution of mass in an isothermal gas sphere [19]. We will proceed as in the preceding section and solve this equation for certain cases, where $k = 1, 2, 3$ and 4.

Case 1: $k = 1$

To solve the Lane–Emden equation (31) by using the correction functional reads

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(t) + \int_0^t \frac{r}{t} e^{y_0(r)} dr \right) dt, \quad y(0) = 0, \tag{32}$$

Proceeding as before, using the truncated Taylor expansion of e^y as

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \frac{1}{4!}y^4, \tag{33}$$

we obtain the following approximations

$$\begin{aligned}
y_0(x) &= 0,\\
y_1(x) &= -\frac{1}{2!}x^2,\\
y_2(x) &= -\frac{1}{2!}x^2 + \frac{1}{64}x^4 + \cdots,\\
y_3(x) &= -\frac{1}{2!}x^2 + \frac{1}{64}x^4 - \frac{1}{768}x^6 + \cdots,\\
y_4(x) &= -\frac{1}{2!}x^2 + \frac{1}{64}x^4 - \frac{1}{768}x^6 + \frac{17}{147456}x^8 + \cdots,\\
y_5(x) &= -\frac{1}{2!}x^2 + \frac{1}{64}x^4 - \frac{1}{768}x^6 + \frac{1}{8192}x^8 + \cdots,\\
&\vdots
\end{aligned} \tag{34}$$

The series solution is therefore given by

$$y(x) = -\frac{1}{2!}x^2 + \frac{1}{64}x^4 - \frac{1}{768}x^6 + \frac{1}{8192}x^8 + \cdots. \tag{35}$$

We now proceed to solve Lane–Emden equation of the second kind for $k = 2$.

Case 2: $k = 2$

For $k = 2$, the correction functional reads

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(t) + \int_0^t \frac{r^2}{t^3} e^{y_0(r)} \right) dt, \quad y(0) = 0. \tag{36}$$

Proceeding as in the previous case, we obtain the following components

$$\begin{aligned}
y_0(x) &= 0, \\
y_1(x) &= -\frac{1}{3!}x^2, \\
y_2(x) &= -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots, \\
y_3(x) &= -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{8}{3 \times 7!}x^6 + \cdots, \\
y_4(x) &= -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{8}{3 \times 7!}x^6 + \frac{122}{9 \times 9!}x^8, \\
&\vdots
\end{aligned} \tag{37}$$

The series solution is therefore given by

$$y(x) = -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{8}{3 \times 7!}x^6 + \frac{122}{9 \times 9!}x^8 + \cdots. \tag{38}$$

For the Lane–Emden equation of the second kind for $k = 3$ and $k = 4$ we obtained the following approximations

$$y(x) = -\frac{1}{8}x^2 + \frac{1}{192}x^4 - \frac{5}{18432}x^6 + \frac{23}{1474560}x^8 + \cdots, \tag{39}$$

and

$$y(x) = -\frac{1}{10}x^2 + \frac{1}{280}x^4 - \frac{1}{6300}x^6 + \frac{43}{5544000}x^8 + \cdots. \tag{40}$$

respectively. We can readily observe that the power series approximations are the same concerning the exponents of x but differ only in the magnitude of the coefficients which are dependent on the shape factor k.

Example 3 We next consider the Lane–Emden-type equation given by

$$y'' + \frac{8}{x}y' + (18y + 4y \ln y) = 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{41}$$

The Volterra integro-differential form for this equation is

$$y'(x) + \int_0^x \frac{t^8}{x^8}\,(18y(t) + 4y(t)\ln(y(t)))\,dt = 0. \tag{42}$$

The correction functional reads

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(t) + \int_0^t \frac{r^8}{t^8}\,(18y_n(r) + 4y_n(r)\ln(y_n(r)))\,dr \right) dt,$$
$$y(0) = 1. \tag{43}$$

By choosing $y_0(x) = 1$, we obtain the following approximations

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1 - x^2,\\
y_2(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{39}x^6 - \frac{1}{180}x^8 + \cdots,\\
y_3(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{37}{2340}x^8 + \cdots,\\
y_4(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots,\\
&\vdots
\end{aligned} \tag{44}$$

The series solution is therefore given by

$$y(x) = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots, \tag{45}$$

and this gives the closed form solution

$$y(x) = e^{-x^2}. \tag{46}$$

Example 4 We next consider the Lane–Emden-type equation given by

$$y'' + \frac{1}{x}y' + (3y^5 - y^3) = 0,\ y(0) = 1,\ y'(0) = 0. \tag{47}$$

The Volterra integro-differential form for this equation is

$$y'(x) + \int_0^x \frac{t}{x}\,(3y^5(t) - y^3(t))\,dt = 0. \tag{48}$$

The correction functional for the equation reads

$$y_{n+1}(x)=y_n(x)-\int_0^x\left(y_n'(t)+\int_0^t\frac{r}{t}\,(3y_n^5(r)-y_n^3(r))\,dr\right)dt,\ y(0)=1, \tag{49}$$

By choosing $y_0(x)=1$, we obtain the following approximations

$$\begin{aligned} y_0(x) &= 1,\\ y_1(x) &= 1-\frac{1}{2}x^2,\\ y_2(x) &= 1-\frac{1}{2}x^2+\frac{3}{8}x^4-\frac{3}{16}x^6+\frac{29}{512}x^8+\cdots,\\ y_3(x) &= 1-\frac{1}{2}x^2+\frac{3}{8}x^4-\frac{5}{16}x^6+\frac{35}{128}x^8+\cdots,\\ &\ \ \vdots \end{aligned} \tag{50}$$

The series solution is therefore given by

$$y(x)=1-\frac{1}{2}x^2+\frac{3}{8}x^4-\frac{5}{16}x^6+\frac{35}{128}x^8+\cdots, \tag{51}$$

and this gives the closed form solution

$$y(x)=\frac{1}{\sqrt{1+x^2}}. \tag{52}$$

Example 5 We finally study the system of nonlinear Lane–Emden equations

$$\begin{aligned} u''+\frac{8}{x}u'+(18u-4u\ln v)&=0,\\ v''+\frac{4}{x}v'+(4v\ln u-10v)&=0, \end{aligned} \tag{53}$$

with initial conditions

$$\begin{aligned} u(0)=1,\quad u'(0)=0,\\ v(0)=1,\quad v'(0)=0, \end{aligned} \tag{54}$$

where the shape factors are $k_1=8$ and $k_2=4$.

The system of Volterra integro-differential forms of this system is given by

$$u'(x) + \int_0^x \frac{t^8}{x^8} (18u(t) - 4u(t)\ln v(t))\, dt = 0,$$
$$v'(x) + \int_0^x \frac{t^4}{x^4} (4v(t)\ln u(t) - 10v(t))\, dt = 0. \tag{55}$$

Consequently, the system of correction functionals for this system of Volterra integro-differential equations is The correction functional for the equation reads

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) + \int_0^t \frac{r^8}{t^8} (18u_n(r) - 4u_n(r)\ln v_n(r))\, dr \right) dt,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left(v_n'(t) + \int_0^t \frac{r^4}{t^4} (4v_n(r)\ln u_n(r) - 10v_n(r)(r))\, dr \right) dt. \tag{56}$$

$$\begin{aligned}
&u_0(x) = 1,\ v_0(x) = 1,\\
&u_1(x) = 1 - x^2,\ v_1(x) = 1 + x^2,\\
&u_2(x) = 1 - x^2 + \frac{1}{2!}x^4 + \cdots,\ v_2(x) = 1 + x^2 + \frac{1}{3!}x^4 + \cdots,\\
&u_3(x) = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \cdots,\ v_3(x) = 1 + x^2 + \frac{1}{3!}x^4 + \frac{1}{3}x^6 + \cdots,\\
&u_4(x) = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots,\\
&\qquad v_4(x) = 1 + x^2 + \frac{1}{3!}x^4 + \frac{1}{3}x^6 + \frac{1}{4!}x^8 + \cdots,\\
&\qquad \vdots,
\end{aligned} \tag{57}$$

where truncated Taylor series for $\ln u$ and $\ln v$ are used to compute the calculations. This gives the exact solution of (53) by

$$(u(x), v(x)) = (e^{-x^2}, e^{x^2}). \tag{58}$$

5 Conclusion

In this work, we established Volterra integro-differential equations equivalent to the Lane–Emden equations of the first and the second kind. The proposed Volterra integro-differential form facilitates the computational work and overcomes the difficulty of the singular behavior at $x = 0$. Moreover, we used only one Lagrange multiplier $\lambda = -1$

and this also minimized the computational work when compared to our work in [7]. The Volterra integro-differential form works for all shape factors $k \geq 1$. The results which we obtained, demonstrate the reliability of the established Volterra integro-differential equations. Numerical examples were examined to confirm the enhancements made by the established equations.

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